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Surface Waves on a Reactive Half Plane

JULIUS KANE

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Surface Waves on a Reactive Half Plane

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Abstract

The following two-dimensional problem is considered: An arbitrary antenna is located at the origin which launches surface waves on a reactive half-plane $y = 0$, $x < 0$. It is assumed that the other half-plane, $y = 0$, $x > 0$ is occupied by a perfect conductor. The antenna is approximated by a distribution of multipoles easily found from the antenna's radiation pattern. With this approximation, simple algebraic expressions are obtained for the ratio of the power delivered to the surface waves as compared to the power in the radiated cylindrical field.

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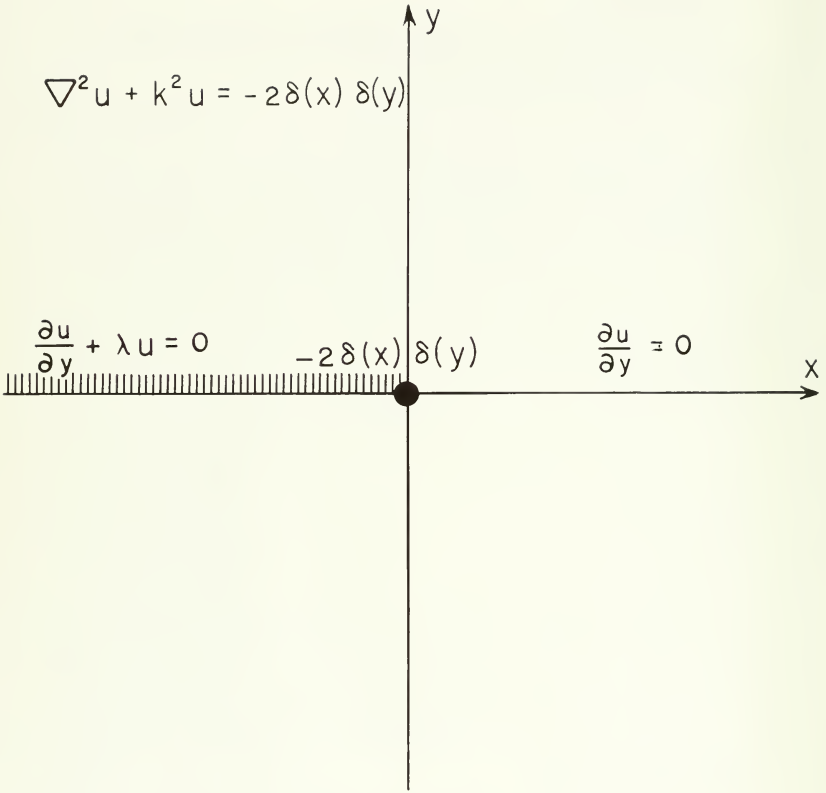

y
$\nabla^2 u + k^2 u = -2\delta(x)\,\delta(y)$
$\frac{\partial u}{\partial y} + \lambda u = 0$
$-2\delta(x)\,\delta(y)$
$\frac{\partial u}{\partial y} = 0$
x

Figure 1

2. Formulation and Solution

The problem (cf. Fig. 1) we shall solve is posed by the three equations:

$$(1) \qquad (\nabla^2 + k^2)u(x,y) = -2\delta(x)\delta(y), \qquad \text{all } x,\ y \geq 0$$

$$(2) \qquad \frac{\partial u}{\partial y} + \lambda u = 0, \qquad x < 0,\ y = 0$$

$$(3) \qquad \frac{\partial u}{\partial y} = 0, \qquad x > 0,\ y = 0$$

where $u(x,y)$ represents the transverse magnetic component H_z, $H_x = H_y = 0$. For convenience k^2 has a small positive imaginary part $k = |k|e^{i\delta}$, and a time factor $e^{-i\omega t}$ is surpressed. Owing to the normalization* chosen for the source term, the free space Green's function of (1) is

$$(4) \qquad \frac{i}{2} H_o^{(1)}(k\rho) = \frac{1}{2\pi i} \int_C \frac{e^{i\nu x + i\sqrt{k^2-\nu^2}\,|y|}}{\sqrt{k^2 - \nu^2}}\, d\nu$$

where the contour C is taken along the real axis in the ν-plane as shown in Figure 2. Since (4) already has the correct singularity at the origin, it follows that if we seek a solution of the form

$$(5) \qquad u(x,y) = \frac{1}{2\pi i} \int_C A^+(\nu)\, \frac{e^{i\nu x + i\sqrt{k^2-\nu^2}}}{\sqrt{k^2 - \nu^2}}\, d\nu$$

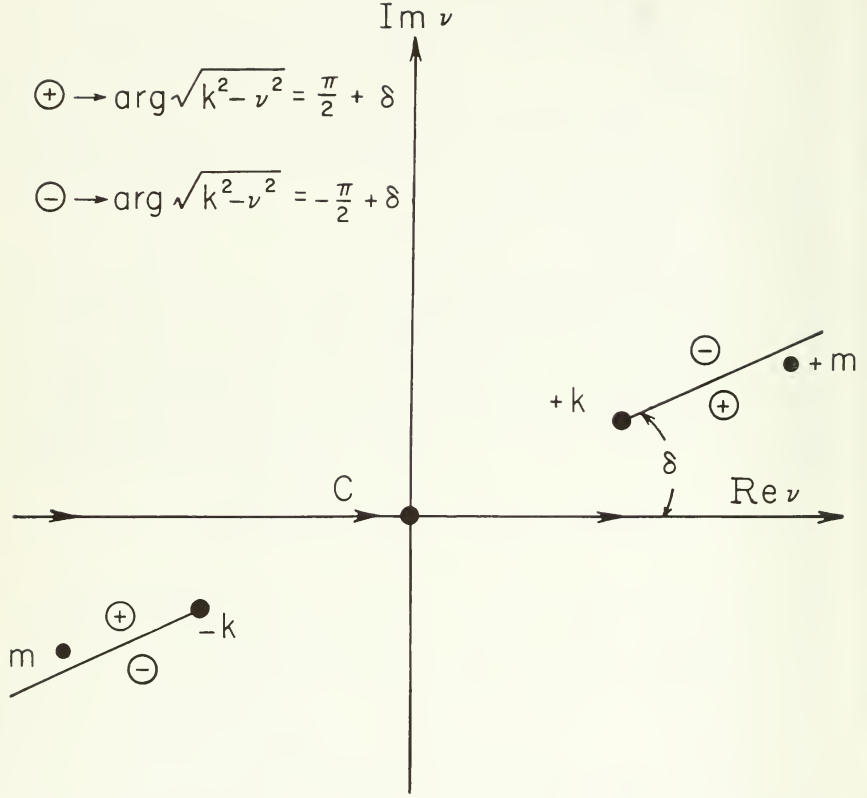
where

$$(6) \qquad A^+(\nu) = \mathcal{O}(1) \text{ as } |\nu| \to \infty\ ,\ \operatorname{Im} \nu \geq 0$$

then equation (1) will continue to be satisfied. At the boundary we

---

* Recall that the usual source strength of $-4\pi\delta(x)\delta(y)$ leads to the Green's function $\pi i\, H_o^{(1)}(k\rho)$.



The magnitude of δ is greatly exaggerated for purposes of clarity

Figure 2: Complex ν plane

assume that we can differentiate freely under the integral sign, and then if $A^+(\nu)$ is an analytic function in the upper ν-plane the boundary condition (3) will be satisfied since we can close the contour C for $x > 0$. Again, it is sufficient that if $G^-(\nu)$ defined by

$$(7) \qquad G^-(\nu) = A^+(\nu) \left[1 + \frac{\lambda}{i\sqrt{k^2 - \nu^2}} \right]$$

is an analytic function in the lower ν-plane and $\mathcal{O}(1)$ as $|\nu| \to \infty$, Im $\nu \leq 0$ the boundary condition (2) will be satisfied. Kay [1], and Bazer and Karp [2] show that the function $\sigma(\nu)$ can be expressed as

$$(8) \qquad \sigma(\nu) = \left[1 + \frac{\lambda}{i\sqrt{k^2 - \nu^2}} \right] = \frac{\sigma^+(\nu)}{\sigma^-(\nu)}$$

where $\sigma^+(\nu)$ is analytic in the upper ν-plane Im $\nu \geq$ Im k and $\sigma^-(\nu)$ is analytic for Im $\nu \leq$ Im k. It follows that equation (7) can be written as

$$(9) \qquad G^-(\nu)\ \sigma^-(\nu) = A^+(\nu)\ \sigma^+(\nu).$$

It can be shown* that $|\sigma^+(\nu)|$ and $|\sigma^-(\nu)|$ are bounded above and below by positive constants in their respective domains of analyticity. Owing to the postulated behavior of $G^-(\nu)$ and $A^+(\nu)$, the left side of (9) is bounded and analytic for Im $\nu \leq 0$ and the right side is bounded and analytic for Im $\nu \geq 0$ implying that each side is the analytic continuation of the other. If we invoke the Liouville theorem we discover that each side of (9) is equal to some constant K proportional to the strength of the source, whence

---

* For example reference [2].

$$G^-(\nu)\ \sigma^-(\nu) = A^+(\nu)\ \sigma^+(\nu) = +K$$

or

$$(10)\qquad \begin{cases} A^+(\nu) = \dfrac{K}{\sigma^+(\nu)} \\ \\ G^-(\nu) = \dfrac{K}{\sigma^-(\nu)} \end{cases} .$$

Observe that (10) yields kernels $A^+(\nu)$ and $G^-(\nu)$ having the required asymptotic behavior for large $|\nu|$.

The integral representation of the solution (5) is then

$$(11)\qquad u(x,y) = \frac{K}{2\pi i} \int_C \frac{1}{\sigma^+(\nu)} \frac{e^{i\nu x + i\sqrt{k^2-\nu^2}\, y}}{\sqrt{k^2 - \nu^2}}\, d\nu$$

or alternately by (8)

$$(12)\qquad u(x,y) = \frac{K}{2\pi i} \int_C \frac{1}{\sigma^-(\nu)\sigma(\nu)} \frac{e^{i\nu x + i\sqrt{k^2-\nu^2}\, y}}{\sqrt{k^2 - \nu^2}}\, d\nu .$$

From (11) we observe that the poles of the integrand are the zeros of $\sigma^+(\nu)$. Since $|\sigma^+(\nu)|$ is bounded above and below in the upper half plane, these can only arise for Im $\nu \leq$ Im k. For reference we locate the poles of the integrand of (11).

We calculate that

$$\frac{1}{\sigma^+(\nu)} = \frac{1}{\sigma^-(\nu)\sigma(\nu)} = \frac{1}{\sigma^-(\nu)} \quad \frac{i\sqrt{k^2 - \nu^2}}{i\sqrt{k^2 - \nu^2} + \lambda}$$

$$= \frac{1}{\sigma^-(\nu)} \quad \frac{i\sqrt{k^2 - \nu^2}\,(\lambda - i\sqrt{k^2 - \nu^2})}{(\sqrt{\lambda^2 + k^2} - \nu)(\sqrt{\lambda^2 + k^2} + \nu)} \quad .$$

The only pole that can arise is in the lower ν-plane at ν_o given by

(13) $$\nu_o = -\sqrt{k^2 + \lambda^2} \quad .$$

The corresponding residue of (11) is found to be

(14) $$\frac{K}{\sigma^-(-\sqrt{k^2 + \lambda^2})} \quad \frac{\lambda}{\sqrt{\lambda^2 + k^2}} \quad e^{-i\sqrt{k^2 + \lambda^2}\,x - \lambda y}$$

which represents the surface wave contribution.

3. The Far Field

Introduce the transformations

(1) $$\nu = k \cos \varphi$$

and

(2) $$x = \rho \cos \vartheta, \quad y = \rho \sin \vartheta$$

so that (2.11) becomes

$$(3) \qquad u(\rho,\vartheta) = -\frac{K}{2\pi i}\int_P \frac{1}{+(k\cos\varphi)}\, e^{ik\rho\cos(\varphi-\vartheta)}\, d\varphi$$

in which the contour C has been transformed into the path P (Fig. 3). If we transform the alternate representation (2.12) to the φ-plane, we obtain

$$(4) \qquad u(\rho,\vartheta) = -\frac{K}{2\pi i}\int_P \frac{ik\sin\varphi\, e^{ik\rho\cos(\varphi-\vartheta)}\, d\varphi}{\sigma^-(k\cos\varphi)(ik\sin\varphi+\lambda)}$$

from which it is seen that the pole at ν_o which generates the surface wave now arises for φ_o given by

$$(5) \qquad \sin\varphi_o = -\frac{\lambda}{ik} \text{ or } \cos\varphi_o = -\sqrt{1+(\lambda/k)^2}\ .$$

An asymptotic development of either (3.3) or its equivalent (3.4) is readily obtained by deforming the path P into the steepest descents contour $S(\vartheta)$. For a given ϑ this contour is defined as the locus of points in the φ-plane which satisfy the equation

$$(6) \qquad \cos(\varphi - \vartheta) = 1 + i\, s^2$$

with s as a real parameter. It is easily established that $S(\vartheta)$ originates at Re $\varphi = \vartheta - \pi/2$, Im $\varphi = \infty$, passes through the saddle point $\varphi = \vartheta$ on the real axis, and terminates at infinity Re $\varphi = \vartheta + \pi/2$, Im $\varphi = -\infty$. Inspection of Figure 3 reveals that for $0 \le \vartheta < \pi/2$ we can deform C freely to $S(\vartheta)$ without picking up the residue at φ_o, and for some ϑ in

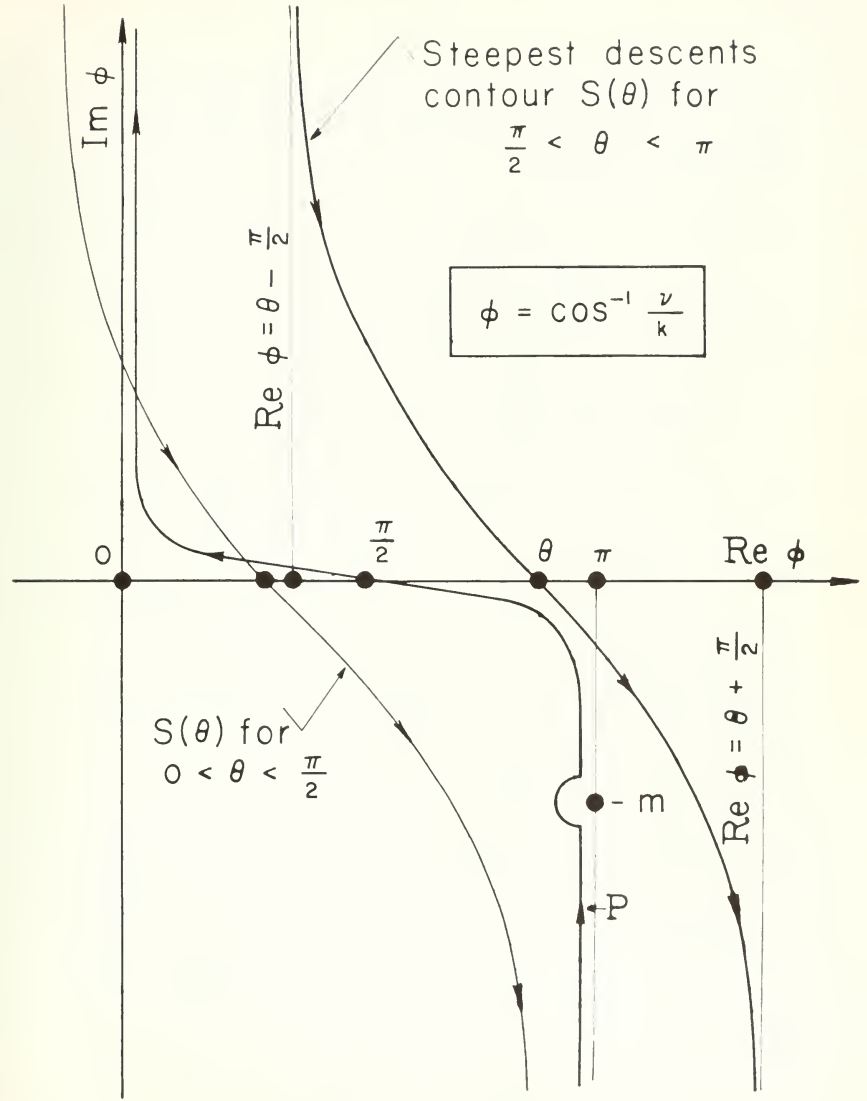


Figure 3: Complex ϕ plane

the interval $\pi/2 < \vartheta \leq \pi$ we pick up the residue. With the aid of (5) and (6) it is established by comparing the real parts of (3.6) that this critical angle ϑ_c is given by

$$\vartheta_c = \cos^{-1}\left(\frac{1}{\sqrt{1 + (\lambda/k)^2}}\right) \tag{7}$$

where $\pi/2 \leq \vartheta_c \leq \pi$ so that we only obtain the surface wave on the left for $x < 0$.

For $k\rho \gg 1$, we then obtain the far-field behavior

$$\lim_{\rho \to \infty} u(\rho,\vartheta) = K\left[\frac{e^{-imx-\lambda y}}{\sigma^-(-m)\ m}\, U(\vartheta-\vartheta_c) + \frac{e^{ik(\rho + \pi/4)}}{\sigma^+(k\cos\vartheta)\ \sqrt{2\pi k}}\right] + \mathcal{O}\left[(k\rho)^{-3/2}\right] \tag{8}$$

where

$$m = \sqrt{k^2 + \lambda^2} \tag{9}$$

and $U(\vartheta)$ is the Heaviside unit function. The result (3.8) should be modified somewhat, if $\lambda \sim 0$, to allow for a pole in the vicinity of the saddle point for $\vartheta = \pi$. This detail is fully discussed in [2] and [4] where the procedure for isolating the pole is explained.

Apart from terms $\mathcal{O}\left[(k\rho)^{-3/2}\right]$, the radiation field contains a cylindrical wave $p(\vartheta)e^{ik\rho}/\sqrt{k\rho}$ with a pattern function $p(\vartheta)$ proportional* to

$$\frac{1}{\sigma^+(k\ \cos\ \vartheta)} = \sigma^-(-\ k\ \cos\ \vartheta), \tag{10}$$

and in addition a surface wave field whose amplitude is proportional to

$$\frac{1}{\sigma^-(-m)}\ . \tag{11}$$

---

* Since $\sigma(\nu)$ is even in ν, we have $\sigma^+(\nu) = 1/\sigma^-(-\nu)$, [2].

For each field the power carried off to infinity varies as the absolute square of these coefficients. As Kay and Lurye have observed, it is an extraordinary occurrence in that the absolute squares of (3.10) and (3.11) are simple elementary functions. From Kay's paper, equation (66), we obtain in our notation:

$$(12) \qquad \left|\sigma^-(-k\cos\vartheta)\right|^2 = \frac{\cos\vartheta + 1}{\cos\vartheta + \sqrt{1 + (\lambda/k)^2}} \qquad 0 \le \vartheta \le \pi$$

$$(13) \qquad \left|\sigma^-(-m)\right|^2 = \left|\sigma^-(-\sqrt{k^2+\lambda^2})\right|^2 = \frac{\sqrt{k^2+\lambda^2}+k}{2\sqrt{k^2+\lambda^2}} = \frac{m+k}{2m} .$$

Note that our λ corresponds to Kay's a.

4. The Energy Distribution

In this section we shall obtain an explicit expression for the ratio of the power delivered to the surface wave field as compared to the cylindrical radiation field. From (3.8) we see that the surface wave is

$$(1) \qquad u_{surf}(x,y) = \frac{\lambda K}{m\,\sigma^-(-m)}\; e^{-i\sqrt{k^2+\nu^2}\,x - \lambda y}\; U(\vartheta - \vartheta_c).$$

The Poynting flux $\prod$ transported by this field is readily calculated if we recall that $u(x,y)$ represents the transverse magnetic component H_z.

$$\prod_{\text{surf}} = \int_0^\infty \left[E_y \cdot H_z^*\right]_{\text{surf}} dy = \frac{1}{i\omega\epsilon} \int_0^\infty \left[\frac{\partial u}{\partial x} u^*\right]_{\text{surf}} dy$$

$$= \frac{\lambda}{2m\omega\epsilon} \left|\frac{K}{\sigma^-(-m)}\right|^2 .$$

If we use Kay's result (3.13) then

$$(2) \qquad \prod_{\text{surf}} = \frac{\Lambda |K|^2}{\omega\epsilon\left(\sqrt{1+\Lambda^2}+1\right)} = \tanh\left(\frac{1}{2}\sinh\Lambda\right)$$

where $\Lambda = \lambda/k$.

In the same fashion we can compute the radiated power of the cylindrical field by using (2.12)

$$(3) \qquad u_{cyl} = \frac{K}{\sigma^+(k\cos\vartheta)\sqrt{2\pi k\rho}} e^{i(k\rho+\pi/4)} + \mathcal{O}\left[(k\rho)^{-3/2}\right]$$

whence

$$\prod_{\text{cyl}} = \int_0^\pi \left[E_\vartheta \cdot H_z^*\right]_{\text{cyl}} \rho d\vartheta$$

$$= \frac{1}{i\omega\epsilon} \int_0^\pi \left[u_\rho u_z^*\right]_{\text{cyl}} \rho d\vartheta$$

$$= \frac{|K|^2}{2\pi\omega\epsilon} \int_0^\pi \frac{d\vartheta}{|\sigma^+(k\cos\vartheta)|^2} = \frac{|K|^2}{2\pi\omega\epsilon} \int_0^\pi \frac{\cos\vartheta + 1}{\cos\vartheta + \sqrt{1+(\lambda/k)^2}} d\vartheta .$$

The integral that appears in this expression is evaluated in Appendix A; we find

$$(4) \qquad \prod_{cyl} = \frac{|K|^2}{2\omega\epsilon} \left[2 + \frac{\Lambda - \sqrt{1+\Lambda^2} + 1}{\Lambda(\Lambda - \sqrt{1+\Lambda^2}} \right].$$

The ratio $\prod_{surf} / \prod_{cyl}$ represents the efficiency of launching the surface wave. With the aid of (2.4) and (4.4) we obtain

$$(5) \qquad \frac{\prod_{surf}}{\prod_{cyl}} = \frac{\Lambda}{(1 + \sqrt{1+\Lambda^2}} \left[\frac{\Lambda(\Lambda - \sqrt{1+\Lambda^2})}{2\Lambda(\Lambda - \sqrt{1+\Lambda^2} + \Lambda - \sqrt{1+\Lambda^2} + 1} \right]$$

For $\Lambda \gg 1$, the bracketed exrpession behaves like 2Λ so that

$$(6) \qquad \lim_{\Lambda \to \infty} \frac{\prod_{surf}}{\prod_{cyl}} = O(\Lambda).$$

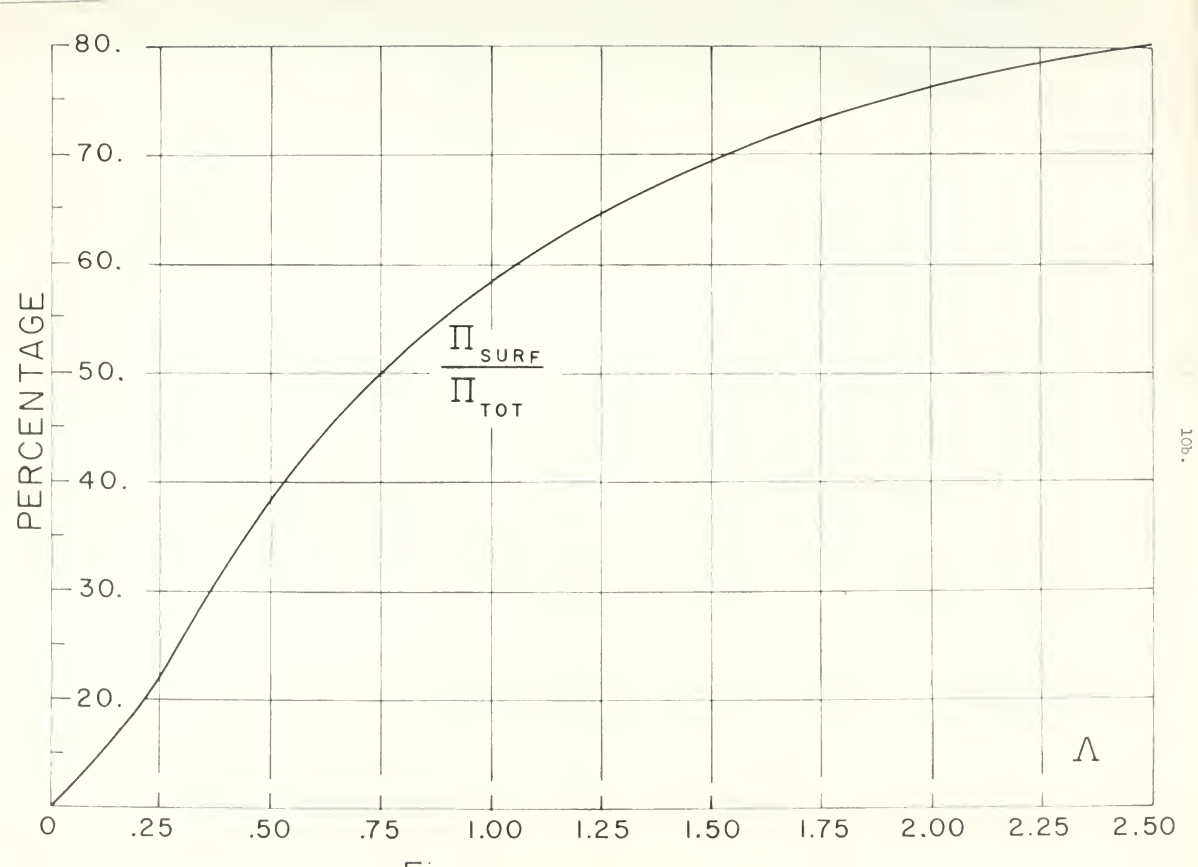
Consequently, even without using a directive source the major portion of the radiated energy is apportioned to the surface wave as we increase the surface reactance Λ.

It is perhaps of greater use to compare the surface wave energy to the total radiated energy

$$\prod_{total} = \prod_{cyl} + \prod_{surf}.$$

If we use (2.4) and (4.4) a little manipulation reveals

Λ	Π_{surf} / Π_{tot}	Π_{cyl} / Π_{tot}	Π_{cyl} / Π_{surf}
0.00	0.0000	1.0000	∞
0.02	0.0198	0.9802	49.5050
0.04	0.0392	0.9608	24.5100
0.06	0.0582	0.9418	16.1816
0.08	0.0768	0.9232	12.0200
0.10	0.0950	0.9050	9.5240
0.12	0.1128	0.8872	7.8632
0.14	0.1302	0.8698	6.6777
0.16	0.1473	0.8527	5.7897
0.18	0.1639	0.8361	5.1002
0.20	0.1802	0.8198	4.5495
0.22	0.1961	0.8039	4.0998
0.24	0.2116	0.7884	3.7258
0.26	0.2268	0.7732	3.4101
0.28	0.2415	0.7585	3.1401
0.30	0.2560	0.7440	2.9067
0.32	0.2700	0.7300	2.7031
0.34	0.2838	0.7162	2.5239
0.36	0.2972	0.7028	2.3650
0.38	0.3102	0.6898	2.2234
0.40	0.3230	0.6770	2.0963
0.42	0.3354	0.6646	1.9817
0.44	0.3475	0.6525	1.8779
0.46	0.3592	0.6407	1.7834
0.48	0.3708	0.6292	1.6971
0.50	0.3820	0.6180	1.6180
0.60	0.4338	0.5662	1.3052
0.70	0.4793	0.5207	1.0862
0.80	0.5194	0.4806	0.9254
0.90	0.5546	0.4454	0.8030
1.00	0.5858	0.4142	0.7071
1.25	0.6492	0.3508	0.5403
1.50	0.6972	0.3028	0.4343
1.75	0.7344	0.2656	0.3616
2.00	0.7639	0.2361	0.3091
2.25	0.7878	0.2122	0.2694
2.50	0.8074	0.1926	0.2385
2.75	0.8238	0.1762	0.2138
3.00	0.8377	0.1623	0.1937
4.00	0.8769	0.1231	0.1404
5.00	0.9010	0.0990	0.1099
6.00	0.9172	0.0828	0.0902
8.00	0.9377	0.0623	0.0664
10.00	0.9501	0.0499	0.0525


80.
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PERCENTAGE
$\frac{\Pi_{SURF}}{\Pi_{TOT}}$
Λ
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.50
.75
1.00
1.25
1.50
1.75
2.00
2.25
2.50

$$(7) \qquad \prod_{\text{total}} = \frac{|K|^2}{\omega\epsilon} \left[\frac{\Lambda}{M+1} + \frac{\Lambda - M + 1}{2\Lambda(\Lambda - M)} + 1 \right]$$

where $M = \sqrt{1 + \Lambda^2}$ so that

$$(8) \qquad \frac{\prod_{\text{surf}}}{\prod_{\text{total}}} = \frac{\Lambda}{1 + \sqrt{1 + \Lambda^2}} \left[\frac{\Lambda}{M+1} + \frac{\Lambda - M + 1}{2\Lambda(\Lambda - M)} + 1 \right]$$

which is plotted in Figure 4 as a function Λ.

5. Arbitrary Excitation

In actual practice one would not excite a surface wave structure by a line source. We would use a directive antenna A such as an exponential horn to focus the maximum energy into the surface wave. How can we obtain the excitation produced by such a radiator? If we replace the surface wave structure by a metallic plane then the antenna A will produce some radiation pattern $p(\vartheta)$ which will characterize the antenna. The pattern $p(\vartheta)$ is known for $0 \leq \vartheta \leq \pi$ so that it is a simple matter [Appendix B] to approximate it arbitrarily by a cosine polynomial

$$(1) \qquad p(\vartheta) \sim p_N(\vartheta) = \sum_{n=0}^{N} a_n (k \cos \vartheta)^n$$

since the cosine functions $\cos n\vartheta$ or $\cos^n \vartheta$ are complete on the interval $0 \leq \vartheta \leq \pi$. In particular the sine functions $\sin n\vartheta$ are expressible as infinite series of cosine functions for $0 \leq \vartheta \leq \pi$. We observe then that

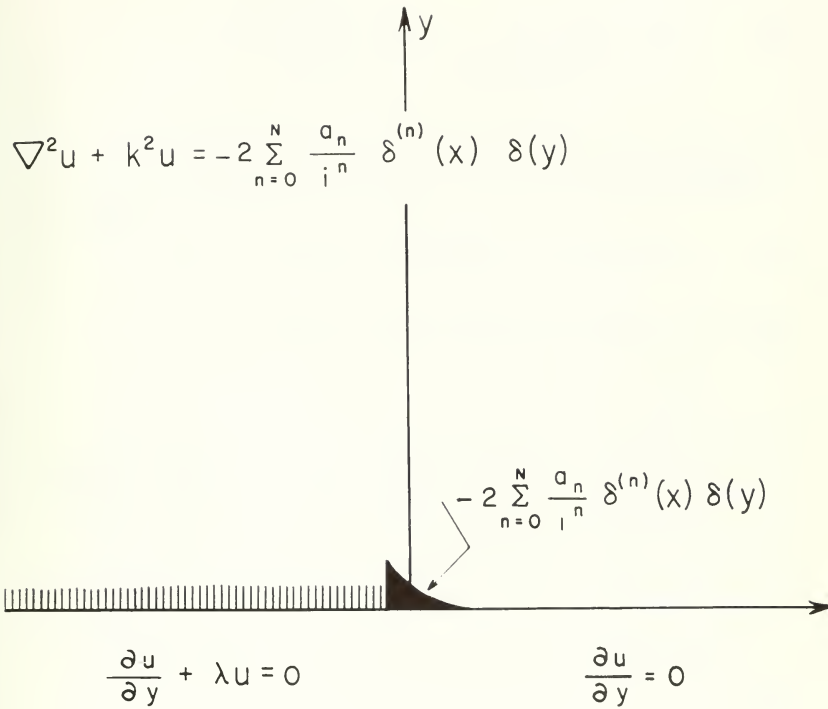
y
$\nabla^2 u + k^2 u = -2 \sum_{n=0}^{N} \frac{a_n}{i^n} \delta^{(n)}(x)\ \delta(y)$
$-2 \sum_{n=0}^{N} \frac{a_n}{i^n} \delta^{(n)}(x)\ \delta(y)$
$\frac{\partial u}{\partial y} + \lambda u = 0$
$\frac{\partial u}{\partial y} = 0$

Figure 5: Approximation to an arbitrary surface wave launcher

$$(2) \qquad u(x,y) = \frac{1}{2\pi i} \int_C \sum_{n=0}^{N} a_n \nu^n \, \frac{e^{i\nu x + i\sqrt{k^2 - \nu^2}\, y}}{\sqrt{k^2 - \nu^2}} \, d\nu$$

is a function which possesses the approximate far-field behavior (1). It is also clear that (2) satisfies the wave equation (cf. Figure 5)

$$(3) \qquad (\nabla^2 + k^2)u(x,y) = -2 \sum_{n=0}^{N} \frac{a_n}{i^n} \delta^{(n)}(x)\delta(y), \qquad y \geq 0$$

where $\delta^{(n)}(x) = \partial^n/\partial x^n \; \delta(x)$ and the boundary condition

$$(4) \qquad \frac{\partial u}{\partial y} = 0 \qquad y = 0.$$

Equation (3) reveals that what we have done is to replace the antenna by an approximate equivalent- a distribution of multipoles. We shall assume that the a_n are real which corresponds to feeding the multipoles in phase intervals of $\pi/2$. One need only solve equation (3) subject to the two part boundary condition

$$(5) \qquad \begin{cases} (a) \quad \dfrac{\partial u}{\partial y} = 0 & y = 0, \; x > 0 \\ (b) \quad \dfrac{\partial u}{\partial y} + \lambda u = 0 & y = 0, \; x < 0 \end{cases}$$

to obtain the effect of introducing the surface wave structure in the left half plane.

Our procedure is to choose

$$u(x,y) = \frac{1}{2\pi i} \int_C A^+(\nu)E(\nu) \frac{e^{i\nu x + i\sqrt{k^2 - \nu^2}\, y}}{\sqrt{k^2 - \nu^2}}\, d\nu \tag{6}$$

where $E(\nu)$ is a known entire function of ν, i.e., a Chebyshev polynomial approximation defined by

$$\begin{cases} E(\nu) = \sum_{n=0}^{N} a_n \nu^n \\ \\ E(k \cos \vartheta) = p_N(\vartheta) \end{cases} \tag{7}$$

If $A^+(\nu)$ is an analytic function of ν for $\operatorname{Im} \nu \geq 0$, the boundary condition (5a) will be satisfied. In addition, if

$$A^+(\nu) = O(1). \qquad |\nu| \to \infty , \tag{8}$$

then (6.5) will have the singularity at the origin demanded by (3.5). For $x < 0$, $y = 0$ we observe that if

$$G^-(\nu) = A^+(\nu)E(\nu) \left[1 + \frac{\lambda}{i\sqrt{k^2 - \nu^2}}\right] \tag{9}$$

is an analytic function of algebraic order in the lower ν-plane, (5b) will be satisfied. If we assume (8) then we see by (9) that

$$G^-(\nu) = O(\nu^n), \qquad |\nu| \to \infty . \tag{10}$$

Use of (2.8) yields by (9)

$$G^{-}(\nu)\ \sigma^{-}(\nu) = A^{+}(\nu)\ E(\nu)\ \sigma^{+}(\nu). \tag{11}$$

Once again we repeat the argument that each side of (11) is analytic in a half plane with a common region (at least the real axis) of analyticity and hence is the analytic continuation of the other. Owing to the assumed growths, we obtain by an appeal to the Liouville theorem that each side is equal to a polynomial $P^{N}(\nu)$ of order N, hence

$$A^{+}(\nu)\ E(\nu)\ \sigma^{+}(\nu) = P^{N}(\nu)$$

or

$$A^{+}(\nu)\ \sigma^{+}(\nu) = \frac{P^{N}(\nu)}{E(\nu)}\ . \tag{12}$$

Since $E(\nu)$ is a polynomial with real coefficients its zeros are either on the real axis or conjugate pairs on the imaginzary axis. Unless $P^{N}(\nu) = K\ E(\nu)$ where K is some constant, the right side of (12) will contradict the analyticity of $A^{+}\sigma^{+}$ for Im $\nu \geq 0$, whence

$$A^{+}(\nu) = \frac{K}{\sigma^{+}(\nu)} \tag{13}$$

or

$$u(x,y) = \frac{K}{2\pi i} \int_{C} \frac{E(\nu)}{\sigma^{+}(\nu)}\ \frac{e^{i\nu x + i\sqrt{k^2 - \nu^2}\, y}}{\sqrt{k^2 - \nu^2}}\ d\nu \tag{14}$$

and again the constant K is proportional to the strength of the source. A straightforward application of the method of saddle points yields

$$(15) \qquad \lim_{\rho \to \infty} u(\rho,\vartheta) = u_{surf}\, U(\vartheta - \vartheta_c) + u_{cyl} + \mathcal{O}\left[(k\rho)^{-3/2}\right]$$

where the surface wave field u_{surf} is

$$(16) \qquad u_{surf}(x,y) = \frac{\lambda K}{m} \frac{E(-m)}{\sigma^{-}(-m)}\, e^{-i\sqrt{k^2+\lambda^2}\, x - \lambda y}$$

and the cylindrical field u_{cyl} is

$$(17) \qquad u_{cyl} = \frac{K\, p_N(\vartheta)}{\sigma^{+}(k \cos \vartheta)} \frac{e^{ik(\rho + \frac{\pi}{4})}}{\sqrt{2\pi k\rho}}$$

The power pattern $P(\vartheta)$ of the cylindrical field is distributed as a function of ϑ by

$$(18) \qquad P(\vartheta) = |p_N(\vartheta)|^2 \frac{\cos \vartheta + 1}{\cos \vartheta + \sqrt{1 + (\lambda/k)^2}}\,.$$

It is interesting to observe that <u>this expression is $|p_N(\vartheta)|^2$ modulated by the previous power pattern function of the line source.</u>

The Poynting flux $\prod$ transported by each field is readily calculated

$$\prod_{surf} = \frac{1}{i\omega\epsilon} \int_0^\infty \left[\frac{\partial u}{\partial x}\, u^*\right]_{surf} dy = \frac{|K|^2}{\omega\epsilon} \frac{\lambda}{m+k} |E(-m)|^2.$$

From (3.7) we have $E(-m) = p_N(\cos^{-1}\frac{m}{k}) = p_n(\cos^{-1}\sqrt{1+\Lambda^2})$ so that the amplitude of the surface wave is proportional to the reactive power

delivered by the antenna to the complex angle $\vartheta = \cos^{-1}\sqrt{1+\Lambda^2}$.

$$\prod_{surf} = \frac{|K|^2}{\omega\epsilon}\ \frac{\lambda}{m+k}\ |E(-m)|^2. \tag{19}$$

The cylindrical radiated power becomes

$$\prod_{cyl} = \frac{1}{i\omega\epsilon}\int_0^\pi u_\rho u^*\, d\vartheta = \frac{|K|^2}{2\pi\omega\epsilon}\int_0^\pi \left|\frac{p_N(\vartheta)}{+(k\cos\vartheta)}\right|^2 d\vartheta$$

or

$$\prod_{cyl} = \frac{|K|^2}{2\pi\omega\epsilon}\int_0^\pi |p_N(\vartheta)|^2\ \frac{\cos\vartheta+1}{\cos\vartheta+\sqrt{1+\Lambda^2}}\, d\vartheta \tag{20}$$

and then

$$\frac{\prod_{surf}}{\prod_{cyl}} = \frac{2\pi\Lambda}{1+\sqrt{1+\Lambda^2}}\ |E(-m)|^2 \int_0^\pi |p(\vartheta)|^2\ \frac{\cos\vartheta+1}{\cos\vartheta+\sqrt{1+\Lambda^2}}\, d\vartheta \tag{21}$$

In Appendix A we evaluate integrals of the form

$$I_n = \int_0^\pi \cos^n\vartheta\ \frac{\cos\vartheta+1}{\cos\vartheta+\sqrt{1+\Lambda^2}}\, d\vartheta$$

which would materially ease the labor in computing (21) for any specified radiation pattern.

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Appendix A

Evaluation of the integral I where $M = \sqrt{1 + (\frac{\lambda}{k})^2}$

$$I_o = \int_0^{\pi} \frac{\cos \vartheta + 1}{\cos \vartheta + M} \, d\vartheta = \frac{1}{2} \int_0^{2\pi} \frac{\cos \vartheta + 1}{\cos \vartheta + M} \, d\vartheta \, , \qquad |M| > 1.$$

This integral can be evaluated very simply by residues; define

$$z = e^{i\vartheta}$$

whence

$$dz = i \, z \, d\vartheta$$

$$\cos \vartheta = \frac{1}{2}(z + \frac{1}{z})$$

so that I_o becomes

$$I_o = \frac{\pi}{2\pi i} \oint_{|z|=1} \frac{z^2 + 2z + 1}{z^2 + 2Mz + 1} \, \frac{dz}{z} \, .$$

The denominator can be readily factored

$$(z^2 + 2Mz + 1) = (z - \alpha)(z - \beta)$$

where

$$\alpha = - M + \sqrt{M^2 - 1} \, , \quad \beta = - M - \sqrt{M^2 - 1}$$

and since $|M| > 1$, clearly $|\alpha| < 1$, $|\beta| > 1$. If we add the residues at $z = 0$ and $z = \alpha$, we obtain

$$I_o = \pi\left[\frac{\alpha^2 + 2\alpha + 1}{\alpha(\alpha - \beta)} + \frac{1}{\alpha\beta}\right]$$

or in terms of $\Lambda = \lambda/k$

$$I_o = \pi\left[2 + \frac{\Lambda - \sqrt{1 + \Lambda^2} + 1}{\Lambda(\Lambda - \sqrt{1 + \Lambda^2})}\right].$$

It is a simple matter to note that $I_o \to 0$ as $\Lambda \to \infty$, and $I_o \to \pi$ as $\Lambda \to 0$.

If the integral is of the form

$$I_n = \int_o^{\pi} \cos^n\vartheta \ \frac{\cos\vartheta + 1}{\cos\vartheta + M}\, d\vartheta$$

the same procedure yields

$$I_n = \frac{1}{2\pi i}\ \frac{\pi}{2^n} \oint_{|z|=1} \frac{z^2 + 2z + 1}{z^2 + 2Mz + 1}\ \frac{\left(z + \frac{1}{z}\right)^n}{z}\, dz\ .$$

Observe that

$$\frac{1}{z}\left(z + \frac{1}{z}\right)^1 = 1 + \frac{1}{z^2}$$

$$\frac{1}{z}\left(z + \frac{1}{z}\right)^2 = z^2 + \frac{2}{z} + 1$$

and in general for n odd the coefficient of $1/z$ vanishes for this factor;

for n even the coefficient of $1/z$ is

$$ {}^{n}C_{n/2} = \frac{\Gamma(n+1)}{\Gamma^2(\frac{n}{2}+1)} \quad . $$

Hence for n odd, the only pole of the integrand is at $z = \alpha$, we find

$$ I_n = I_o \left(\frac{\Lambda^2 - 2\Lambda\sqrt{1+\Lambda^2} + 1}{\Lambda - \sqrt{1+\Lambda^2}} \right)^n , \qquad \text{. n odd} $$

$$ I_n = I_o \left(\frac{\Lambda^2 - 2\Lambda\sqrt{1+\Lambda^2} + 1}{\Lambda - \sqrt{1+\Lambda^2}} \right)^n + \frac{\pi\Gamma(n+1)}{\Gamma^2(\frac{n+2}{2})} , \qquad \text{n even .} $$

Appendix B

First, observe that the cos $n\vartheta$ functions form a complete set on the interval $0 \leq \vartheta \leq \pi$. Expand $p(\vartheta)$ as a Fourier cosine series

$$(1) \qquad p(\vartheta) = \sum_{n=0}^{\infty} a_n \cos n\vartheta$$

in the interval $0 \leq \vartheta \geq \pi$ either from an analytic knowledge of $p(\vartheta)$ or numerically [3] from a measured radiation pattern. Truncate the series at some $n = N$ and express each cos $n\vartheta$ as a polynomial of order n in cos ϑ, e.g.

$$\cos 4\vartheta = 8 \cos^4\vartheta - 8 \cos^2\vartheta + 1.$$

Such a collection of terms in (1) will yield

$$(2) \qquad p(\vartheta) = \sum_{n=0}^{N} a_n \cos^n\vartheta$$

or as a function of $\nu = k \cos \vartheta$

$$(3) \qquad E_N(\nu) = \sum_{n=0}^{N} a_n \left(\frac{\nu}{k}\right)^n .$$

The expression (3) corresponds to a Chebyshev approximation [3] of $E_\infty(\nu)$ in the interval $\left|\frac{\nu}{k}\right| \leq 1$ by a polynomial of order N.

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